In this section, we prove the Bern-Carrasco-Johansen (BCJ) and Kawai-Lawenllen-Tye (KLT) relations from the string theory viewpoint.

String theory is useful computational tool for the real-world non-supersymmetric amplitudes.

In the section, we do not take care of the normalization convention of the amplitude (spacetime metric, Lie algebraic generator, i for the amplitude, ...). The reason is that we are only interested in homogeneous linear relations.

I. OPEN STRING AMPLITUDES AND THE BCJ RELATION

For the open string amplitude with Chan-Paton factors. the tree amplitude is again decomposed to the sum of partial amplitudes,

$$\mathbb{A}^{\text{open}} = \sum_{\sigma \in S^{n-1}} \operatorname{tr}(t^{a_{\sigma_1}} t^{a_{\sigma_2}} \dots t^{a_n}) \mathscr{A}(\sigma_1 \sigma_2 \dots n) \tag{1}$$

where the partial amplitude is a disk boundary integration.

$$\begin{aligned} \mathscr{A}(12\dots n) = & \frac{8ig^{n-2}}{\alpha'^2} \frac{1}{2^{\frac{n-2}{2}}} \int_{x_1 \le x_2 \le \dots x_n} \frac{dx^1 dx^2 \dots dx^n}{dx^a dx^b dx^c} (x_b - x_a)(x_c - x_b)(x_c - x_a) \\ & \times \left(\prod_{1 \le i < j \le n} (x_j - x_i)^{2\alpha' k_i \cdot k_j} \right) \exp \left(\sum_{i<j} \frac{2\alpha' \epsilon_i \cdot \epsilon_j}{(x_i - x_j)^2} + \sum_{i \ne j} \frac{2\alpha' k_i \cdot \epsilon_j}{x_i - x_j} \right) \Bigg|_{\text{linear}} \end{aligned} \tag{2}$$

where x_i's are Kobe-Nielsen variables.

String amplitude is, in general, much more complicated than Yang-Mills amplitude. However, since the string effective action for Bosonic theory is For example, the bosonic string theory's gauge effective theory is,

$$S = \int d^d x \left(-\frac{1}{4g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{2i\alpha'}{3g^2} \operatorname{tr}(F_\mu{}^\nu F_\nu{}^\omega F_\omega{}^\mu) + \dots \right) \tag{3}$$

it is well-known that in the $\alpha' \to 0$ limit,

$$\mathscr{A}(1, \dots n) \to A^{\text{YM}}(1, \dots n) \tag{4}$$

A. 4-point example

We can fix $x_1 = 0$, $x_2 = 1$ and $x_4 = \infty$, and integrate over x_3. The evaluation of (2) is tedious but straightforward. The ordering $\mathscr{A}(1234)$ is

$$\mathscr{A}(1234) = \frac{ig^2}{2} \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(\alpha' u + 1)} K(1234) \tag{5}$$

Where $K(1234)$ is the kinematic factors containing the polarization vectors. And,

$$K(1234) = K^{\text{Type I}}(1234) + K^{\text{Bosonic}}(1234) \tag{6}$$

where $K^{\text{Type I}}$ is the same kinetic term as that in Type I open string theory.

$$\begin{aligned}
K^{\text{Type I}}(1234) \;=\; & \alpha'^2(st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + su\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 + tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4) \\
& -2\alpha'^2 s(\epsilon_1\cdot k_4\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_3\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_3 + \epsilon_1\cdot k_3\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3 \\
& +\epsilon_2\cdot k_4\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_4) \\
& -2\alpha'^2 t(\epsilon_2\cdot k_1\epsilon_4\cdot k_3\epsilon_3\cdot\epsilon_1 + \epsilon_3\cdot k_4\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_4\epsilon_1\cdot k_3\epsilon_3\cdot\epsilon_4 \\
& +\epsilon_3\cdot k_1\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_1) \\
& -2\alpha'^2 u(\epsilon_1\cdot k_2\epsilon_4\cdot k_3\epsilon_3\cdot\epsilon_2 + \epsilon_3\cdot k_4\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4 + \epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_4 \\
& +\epsilon_3\cdot k_2\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_2)
\end{aligned} \tag{7}$$

Supersymmetry puts strong constraints on the possible interactions, so the non-supersymmetric theory have much more terms,

$$\begin{aligned}
K^{\text{Bosonic}}(1234) = {} & 4\alpha'^3 s\Big[\epsilon_1\cdot k_3\epsilon_2\cdot k_3(\epsilon_3\cdot k_1\epsilon_4\cdot k_1 + \epsilon_3\cdot k_2\epsilon_4\cdot k_2) + \\
& \frac{1}{3}(\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1 - \epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1 - \epsilon_4\cdot k_2)\Big] \\
& +4\alpha'^3 t\Big[\epsilon_2\cdot k_1\epsilon_3\cdot k_1(\epsilon_1\cdot k_3\epsilon_4\cdot k_3 + \epsilon_1\cdot k_2\epsilon_4\cdot k_2) + \\
& \frac{1}{3}(\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1)(\epsilon_4\cdot k_3 - \epsilon_4\cdot k_2)\Big] \\
& +\alpha'^3 u[\epsilon_1\cdot k_2\epsilon_3\cdot k_2(\epsilon_2\cdot k_1\epsilon_4\cdot k_1 + \epsilon_2\cdot k_3\epsilon_4\cdot k_3) + \\
& \frac{1}{3}(\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1 - \epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3 - \epsilon_4\cdot k_1)] \\
& +(2\alpha')^2\frac{st}{4}\frac{1}{1+\alpha' u}(\epsilon_1\cdot\epsilon_3 - (2\alpha')\epsilon_1\cdot k_3\epsilon_3\cdot k_1)(\epsilon_2\cdot\epsilon_4 - (2\alpha')\epsilon_2\cdot k_4\epsilon_4\cdot k_2) \\
& +(2\alpha')^2\frac{tu}{4}\frac{1}{1+\alpha' s}(\epsilon_1\cdot\epsilon_2 - (2\alpha')\epsilon_1\cdot k_2\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_4 - (2\alpha')\epsilon_3\cdot k_4\epsilon_4\cdot k_3) \\
& +(2\alpha')^2\frac{su}{4}\frac{1}{1+\alpha' t}(\epsilon_1\cdot\epsilon_4 - (2\alpha')\epsilon_1\cdot k_4\epsilon_4\cdot k_1)(\epsilon_2\cdot\epsilon_3 - (2\alpha')\epsilon_2\cdot k_3\epsilon_3\cdot k_2) \\
& -\alpha'^2(st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - su\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 - tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4)
\end{aligned} \tag{8}$$

It is interesting to look at this amplitude in detail. First, from the Gamma function expansion, the only massless poles are s and t while the u pole is absent. This is consistent with the color-ordered Yang-Mills Feynman diagram analysis.

In the low energy limit, the leading order of the scattering amplitude is,

$$\mathscr{A}(1234) \supset \frac{1}{\alpha'}\frac{1}{st}K^{\text{Type I}} \to A(1234). \tag{9}$$

This term is the same as the Yang-Mills theory scattering amplitude.

Note that, explicitly, both the kinematic factors $K^{\text{Type I}}$ and K^{Bosonic} and totally symmetric in the incoming states, for example, under $1 \leftrightarrow 2$.

$$K^{\text{Type I}}(1234) = K^{\text{Type I}}(2134), \quad K^{\text{Bosonic}}(1234) = K^{\text{Bosonic}}(2134). \tag{10}$$

This permutation exchanges u and t. Hence,

$$\mathscr{A}(1234)\sin(\pi\alpha' t) = \mathscr{A}(2134)\sin(\pi\alpha' u) \tag{11}$$

$$tA(1234) = uA(2134), \tag{12}$$

where the second equality explicitly implies the BCJ relation for four point.

It is not practical to compute the Yang-Mills amplitude for the string theory low-energy limit. However, it is easy to derive Yang-Mills amplitude relations from string theory relations.

Here we drive the 4-point BCJ amplitude from the viewpoint of monodromy relations [1].

$$\mathscr{A}(2134) = \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^2 \int_{-\infty}^{0} dx_2\ (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(x_2) \tag{13}$$

$$\mathscr{A}(1234) = \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^2 \int_{0}^{1} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(x_2) \tag{14}$$

$$\mathscr{A}(1324) = \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^2 \int_{1}^{\infty} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(x_2) \tag{15}$$

where the $\bar{f}(x_2)$ contains the polarizations,

$$\bar{f}(x_2) = \exp\left(\frac{\alpha'}{2}\sum_{i>j}\frac{\zeta_i\cdot\zeta_j}{(x_i-x_j)^2} - \frac{\alpha'}{2}\sum_{i\neq j}\frac{\zeta_i\cdot k_j}{x_i-x_j}\right)\Bigg|_{\text{multiple-linear}}. \tag{16}$$

and we set $x_1 = 0$, $x_3 = 1$ and $x_4 = \infty$. Consider the integral,

$$0 = \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^2 \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(x_2) \tag{17}$$

where ϵ is a small positive constant. The branch cut for the complex function z^a is defined to be the negative real axis. By the careful analysis of (17), we have

$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathscr{A}(2134) + \mathscr{A}(1234) + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathscr{A}(1324) = 0. \tag{18}$$

By stripping the polarization vectors, we can treat the partial amplitudes as real quantities. Then, the real and complex part of the identity read,

$$\begin{aligned}
&\cos\left(\pi(\frac{\alpha'}{2}k_1\cdot k_2)\right)\mathscr{A}(2134)+\mathscr{A}(1234)+\cos\left(\pi(\frac{\alpha'}{2}k_2\cdot k_3)\right)\mathscr{A}(1324)=0\\
&\sin\left(\pi(\frac{\alpha'}{2}k_1\cdot k_2)\right)\mathscr{A}(2134)-\sin\left(\pi(\frac{\alpha'}{2}k_2\cdot k_3)\right)\mathscr{A}(1324)=0
\end{aligned}\tag{19}$$

These are string theory relations. Taking the $\alpha\to 0$ limit (low energy limit), we have

$$\begin{aligned}
&A(2134)+A(1234)+A(1324)=0\\
&sA(2134)=tA(1324)
\end{aligned}\tag{20}$$

The first identity is the Kleiss-Kuijf (KK) relation while the second identity is the BCJ relation for *Yang-Mills* amplitude relation.

Combine KK and BCJ relations, we determined that there is only one linearly independent 4-point partial tree amplitude, $A(1234)$.

B. 5-point

For this case, we set $x_1=0$, $x_3=1$ and $x_5=\infty$. The string partial amplitudes are

$$\begin{aligned}
\mathscr{A}(21345) &= \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^3\int_1^\infty dx_4\int_{-\infty}^0 dx_2\ (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}(x_4-x_2)^{\frac{\alpha'}{2}k_2\cdot k_4}\\
&\quad\times(x_4)^{\frac{\alpha'}{2}k_1\cdot k_4}(x_4-1)^{\frac{\alpha'}{2}k_3\cdot k_4}\bar{f}(x_2,x_4) \qquad (21)\\
\mathscr{A}(12345) &= \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^3\int_1^\infty dx_4\int_0^1 dx_2\ (x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}(x_4-x_2)^{\frac{\alpha'}{2}k_2\cdot k_4}\\
&\quad\times(x_4)^{\frac{\alpha'}{2}k_1\cdot k_4}(x_4-1)^{\frac{\alpha'}{2}k_3\cdot k_4}\bar{f}(x_2,x_4) \qquad (22)\\
\mathscr{A}(13245) &= \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^3\int_1^\infty dx_4\int_1^{x_4} dx_2\ (x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3}(x_4-x_2)^{\frac{\alpha'}{2}k_2\cdot k_4}\\
&\quad\times(x_4)^{\frac{\alpha'}{2}k_1\cdot k_4}(x_4-1)^{\frac{\alpha'}{2}k_3\cdot k_4}\bar{f}(x_2,x_4) \qquad (23)\\
\mathscr{A}(13425) &= \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^3\int_1^\infty dx_4\int_{x_4}^\infty dx_2\ (x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3}(x_2-x_4)^{\frac{\alpha'}{2}k_2\cdot k_4}\\
&\quad\times(x_4)^{\frac{\alpha'}{2}k_1\cdot k_4}(x_4-1)^{\frac{\alpha'}{2}k_3\cdot k_4}\bar{f}(x_2,x_4) \qquad (24)
\end{aligned}$$

Consider the integration

$$0=\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dx_2(x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}(x_4-x_2)^{\frac{\alpha'}{2}k_2\cdot k_4}\tag{25}$$

for the fixed x_4. Then we have the monodromy identity,

$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathscr{A}(21345)+\mathscr{A}(12345)+e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathscr{A}(13245)+e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3+k_2\cdot k_4)}\mathscr{A}(13425)=0.\tag{26}$$

which has the following $\alpha' \to 0$ limit as the BCJ identity,

$$s_{12}A(21345) - (s_{23})A(13245) + (s_{23} - s_{24})A(13425) = 0. \tag{27}$$

Using all KK and BCJ relations for the 5-point partial tree amplitudes, we find that only $A(12345)$ and $A(13245)$ are linearly independent.

II. KLT RELATION

KLT relation is a classic relation for string theory [2]. It has surprising implication for graviton amplitudes in Einstein theory and supergravity.

In general, a closed string tree amplitude read,

$$\begin{aligned}\mathcal{A}(12\ldots n) = g_c^{n-2} 4\pi \left(\frac{2}{\alpha'}\right)^{n+1} \int \frac{d^2z_1 d^2z_2 \ldots d^2z_n}{d^2z_a d^2z_b d^2z_c} |z_b - z_a|^2 |z_c - z_b|^2 |z_c - z_a|^2 \\ \times \left(\prod_{1\le i<j\le n} (z_j - z_i)^{\frac{\alpha'}{2} k_i\cdot k_j}\right) \exp\left(\sum_{i<j} \frac{\alpha' \epsilon_i \cdot \epsilon_j}{2(z_i - z_j)^2} + \sum_{i\neq j} \frac{\alpha' k_i \cdot \epsilon_j}{2(z_i - z_j)}\right) \\ \times \left(\prod_{1\le i<j\le n} (\bar{z}_j - \bar{z}_i)^{\frac{\alpha'}{2} k_i\cdot k_j}\right) \exp\left(\sum_{i<j} \frac{\alpha' \bar{\epsilon}_i \cdot \bar{\epsilon}_j}{2(\bar{z}_i - \bar{z}_j)^2} + \sum_{i\neq j} \frac{\alpha' k_i \cdot \bar{\epsilon}_j}{2(\bar{z}_i - \bar{z}_j)}\right)\Bigg|_{\text{bilinear}}\end{aligned} \tag{28}$$

Note that there is no ordering here. Here the subscript "bilinear" means that only the $\epsilon_{i,\mu}\bar{\epsilon}_{i,\nu}$ terms are kept in the final result and recombined as $\epsilon_{i,\mu}\bar{\epsilon}_{i,\nu} \to e_{i,\mu\nu}$. Note that there is no ordering of the closed string vertices.

The first nontrivial KLT relation is the four point scattering amplitudes. The real and imaginary part of the integral in closed string amplitude, can be treated as two independent real integrals. However, we have to take care of the contours. Let $z_1 = 0$, $z_2 = z$, $z_3 = 1$ and $z_4 \to \infty$, and (28) reads,

$$\mathscr{A}_c(1234) = 4\pi g_c^2 \left(\frac{2}{\alpha'}\right)^5 \int d^2z \; z^{\frac{\alpha'}{2}k_1\cdot k_2} (1-z)^{\frac{\alpha'}{2}k_2\cdot k_3} f(z) \; \bar{z}^{\frac{\alpha'}{2}k_1\cdot k_2} (1-\bar{z})^{\frac{\alpha'}{2}k_2\cdot k_3} \bar{f}(\bar{z}) \tag{29}$$

where $f(z)$ is a holomorphic function which contains the polarization vectors. Similar $\bar{f}(\bar{z})$ is the conjugation of $f(z)$.

$$\begin{aligned} f(z) &= \lim_{z_4\to\infty} z_4^2 \exp\left(\sum_{i<j} \frac{\alpha' \epsilon_i \cdot \epsilon_j}{2(z_i - z_j)^2} + \sum_{i\neq j} \frac{\alpha' k_i \cdot \epsilon_j}{2(z_i - z_j)}\right)\Bigg|_{\text{linear}} \\ \bar{f}(\bar{z}) &= \lim_{\bar{z}_4\to\infty} \bar{z}_4^2 \exp\left(\sum_{i<j} \frac{\alpha' \bar{\epsilon}_i \cdot \bar{\epsilon}_j}{2(\bar{z}_i - \bar{z}_j)^2} + \sum_{i\neq j} \frac{\alpha' k_i \cdot \bar{\epsilon}_j}{2(\bar{z}_i - \bar{z}_j)}\right)\Bigg|_{\text{linear}} \end{aligned} \tag{30}$$

The integral is over the whole complex plane, and both z and $\bar{z}$ are complex. Let $z = x + iy$,

$$\int d^2 z \mapsto 2 \int dx \int dy \tag{31}$$

and the integrand is analytic both in x and y. We would like to consider y on the whole complex plane. Note the possible poles of y are

$$y = ix, i(1-x), i(x-1), -ix \tag{32}$$

which are all on the imaginary axis. Hence we can use the Wick rotation, as Fig.(1). Now y is imaginary (up to the infinitesimal prescription), so both $x+iy$ and $x-iy$ are real. Define $\xi = x+iy$

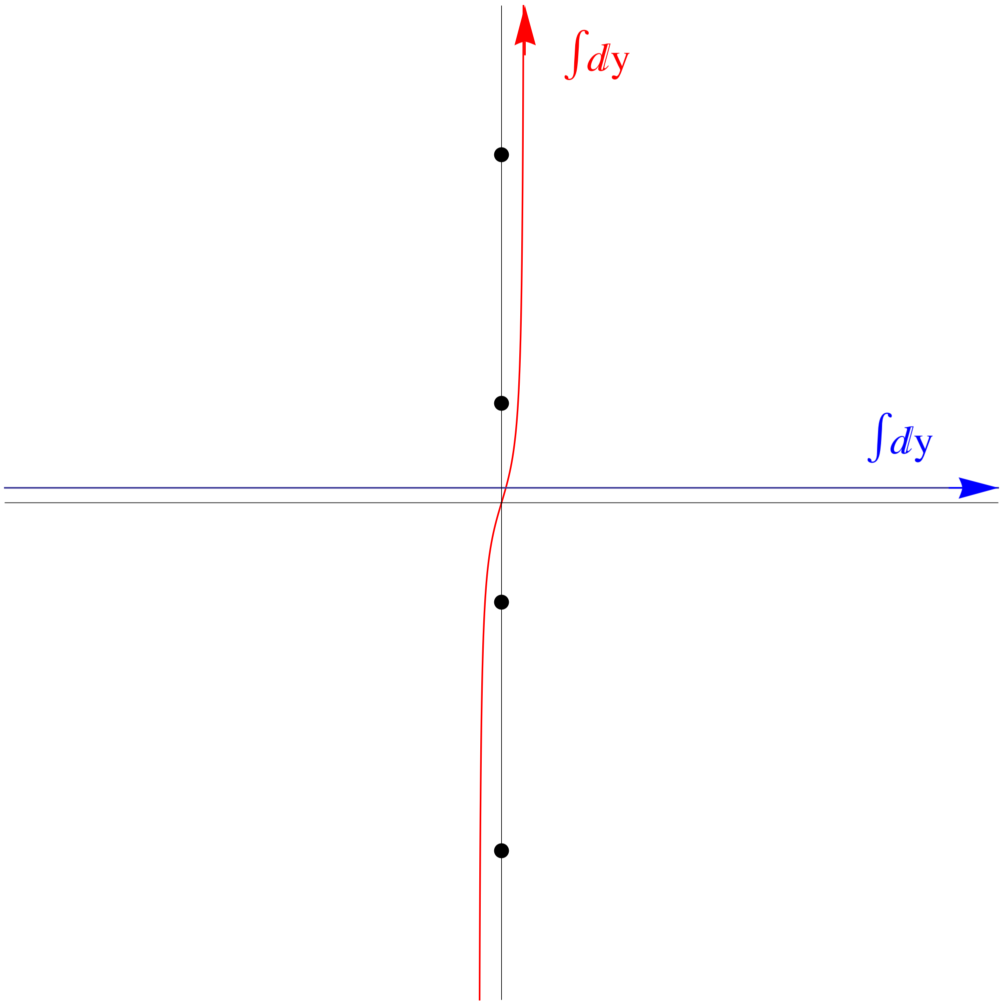

FIG. 1: Analytical continuation of y.

and $\eta = x - iy$, and the integral (28) reads,

$$\mathscr{A}_c(1234) = 4\pi i g_c^2 \left(\frac{2}{\alpha'}\right)^5 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta |\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3} f(\xi)\ |\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)$$
$$\times \exp\left(i\pi\frac{\alpha'}{2}k_1\cdot k_2\theta(-\xi\eta) + i\pi\frac{\alpha'}{2}k_2\cdot k_3\theta(-(1-\xi)(1-\eta))\right) \tag{33}$$

where the phase term comes from the prescription of the contour. $\theta(\ldots)$ is the Heaviside step function. (33) looks like a product of open string amplitudes.

However, there are 3 different orderings in the ξ integral while 3 orderings in the η integral. So it seems that (28) would be the sum of $3\times 3 = 9$ pairs of open string amplitudes. However, we can represent the phase as the contour of η integral. Only when $0<\xi<1$, the contour of η is nontrivial. It is shown in (2). If we use the contour C_1, (33) becomes,

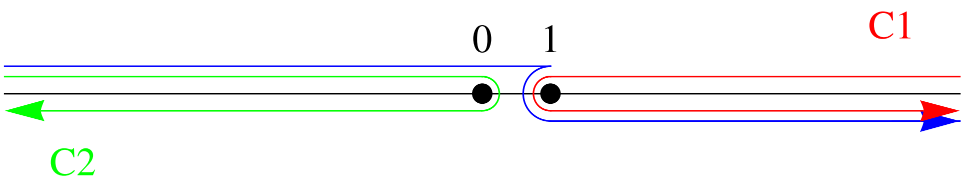

FIG. 2: Contour integral for η when $0<\xi<1$. The original contour can be deformed to either C_1 or C_2.

$$\begin{aligned}\mathscr{A}_c(1234) &= 8\pi g_c^2\left(\frac{2}{\alpha'}\right)^5 \sin\left(\frac{\alpha'\pi k_2\cdot k_3}{2}\right)\times\int_0^1 d\xi|\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3}f(\xi)\\ &\quad\times\int_1^{\infty} d\eta|\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)\\ &= 4\pi\frac{g_c^2}{\alpha' g^4}\sin\left(\frac{\alpha'\pi k_2\cdot k_3}{2}\right)\mathscr{A}(1234)\mathscr{A}(1324)\end{aligned} \tag{34}$$

and similarly if we use the C_2, the result is,

$$\begin{aligned}\mathscr{A}_c(1234) &= 8\pi g_c^2\left(\frac{2}{\alpha'}\right)^5 \sin\left(\frac{\alpha'\pi k_1\cdot k_2}{2}\right)\times\int_0^1 d\xi|\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3}f(\xi)\\ &\quad\times\int_{-\infty}^{0} d\eta|\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)\\ &= 4\pi\frac{g_c^2}{\alpha' g^4}\sin\left(\frac{\alpha'\pi k_1\cdot k_2}{2}\right)\mathscr{A}(1234)\bar{\mathscr{A}}(2134)\end{aligned} \tag{35}$$

These two results are equivalent.

Note that the proof of 4 point KLT relation actually provides all 4 point BCJ relations.

The low energy limit reads,

$$A_{\text{Einstein}}(1234) \propto tA(1234)A(1324) \tag{36}$$

This means the 4-point tree level Einstein gravity amplitude is a double copy of Yang-Mills partial amplitudes, with different orderings.

We may consider gravity MHV amplitude $A_{\text{Einstein}}(1^-2^-3^+4^+)$.

$$A_{\text{Einstein}}(1^-2^-3^+4^+) \propto t\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[34]^4}{[13][32][24][41]} \propto \frac{\langle 12\rangle^4[34]^4}{stu} \tag{37}$$

Note that the pole structure is completely symmetric.

Similarly, n-point ($n > 4$) tree level closed string amplitude can be decomposed as the sum of open string amplitude products. For example, the five-point KLT relation is,

$$\begin{aligned}\mathscr{A}_c(12345) = &\frac{g_c^3}{g^6\alpha'^2}\sin\left(\frac{\alpha'\pi k_1\cdot k_2}{2}\right)\sin\left(\frac{\alpha'\pi k_3\cdot k_4}{2}\right)\mathscr{A}(12345)\bar{\mathscr{A}}(21435)\\ &+\frac{g_c^3}{g^6\alpha'^2}\sin\left(\frac{\alpha'\pi k_1\cdot k_3}{2}\right)\sin\left(\frac{\alpha'\pi k_2\cdot k_4}{2}\right)\mathscr{A}(13245)\bar{\mathscr{A}}(31425).\end{aligned} \tag{38}$$

Note that unlike the four point case, this identity is the sum of two pairs. Taking the low energy limit, we have

$$A_{\text{Einstein}}(12345) \propto s_{12}s_{34}A(12345)\bar{A}(21435) + s_{13}s_{24}A(13245)\bar{A}(31425). \tag{39}$$

The minimum number of pairs for general KLT relations was counted in [2].

$$\begin{aligned}&(n-3)!(\frac{n-3}{2})!(\frac{n-3}{2})!, \quad n \text{ is odd}\\ &(n-3)!(\frac{n-4}{2})!(\frac{n-2}{2})!, \quad n \text{ is even}\end{aligned} \tag{40}$$

A. Hidden identities

The 4D graviton-axion-dilaton gravity action in Einstein frame,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-4\phi}H_{\mu\nu\rho}H^{\mu\nu\rho}) \tag{41}$$

where ϕ is the dilaton and $H_{\mu\nu\rho}$ is the strength of the antisymmetric field. We just keep the two-derivative term and neglect the higher-derivative terms from string theory correction. (So it is the low energy effective action for the gravity sector of any string theory.) The Poincare dual of $H_{\mu\nu\rho}$ is an axion,

$$\partial_\mu b = \frac{1}{6}e^{-4\phi}\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu\rho} \tag{42}$$

where $\epsilon_{0123} = (-\det G)^{1/2}$ and $\epsilon^{0123} = -(-\det G)^{1/2}$. So the action can be rewritten as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{4\phi}\partial_\mu b\partial^\mu b) \tag{43}$$

Combining the axion and the dilaton, we have $S_{\pm} = b \pm ie^{-2\phi}$ and,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}(R - \frac{1}{2}\frac{|\partial_\mu S_+|^2}{(\mathrm{ImS_+})^2}) \tag{44}$$

Note that the axion b is characterized by the shift symmetry

$$b \mapsto b + c \tag{45}$$

where c is a dimensionful constant. Furthermore S_+ has $SL(2,\mathbb{R})$ global symmetry. The group $SL(2,R)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \tag{46}$$

acts on S_+ as,

$$S_+ \mapsto \frac{aS_+ + b}{cS_+ + d}. \tag{47}$$

S_+ takes value in the upper complex plane, which is the moduli space of this theory. We may choose $\langle S_+\rangle = i$. Then the unbroken symmetry is $SO(2)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{48}$$

We will see that this unbroken symmetry lead to the quadratic identities of Yang-Mills theory.

To study the perturbative theory of the axion-dilaton, we may define $S_{\pm} = \sqrt{2}\kappa\tau + i$ and $\tau = \tau_1 + i\tau_2$. Now $\langle\tau\rangle = 0$ and τ has the canonical kinetic terms,

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}R - \frac{1}{2}\int d^4x\sqrt{-G}\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{(1+\sqrt{2}\kappa\tau_2)^2} \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}R - \frac{1}{2}\int d^4x\sqrt{-G}\partial_\mu\tau\partial^\mu\bar{\tau}(1 - 2\sqrt{2}\kappa\tau_2 + 6\kappa^2\tau_2^2 \\ &\quad -8\sqrt{2}\kappa^3\tau_2^3 + \ldots) \end{aligned} \tag{49}$$

all the interaction terms contain positive powers of κ, as it should be. To the leading order, the symmetry 48 acts on τ as,

$$\tau \mapsto \tau + 2i\theta\tau, \tag{50}$$

So τ has an infinitesimal $U(1)$ symmetry. We may think that τ is positively charged and corresponds to the polarization $\epsilon^+\tilde{\epsilon}^-$, while τ is negatively charged and corresponds to the polarization $\epsilon^-\tilde{\epsilon}^+$.

Now we see that the dilaton and the axion combine to form a massless complex scalar field τ, which has a global conserved $U(1)$ charge associated with it. All scattering (tree or loop) amplitudes must obey this charge conservation.

Within helicity amplitudes, graviton j has helicity $\epsilon_j^{\pm}\tilde{\epsilon}_{\tilde{j}}^{\pm} = j^{\pm}\tilde{j}^{\pm}$ and an incoming positively charged scalar field j may be identified with helicity $j^{+}\tilde{j}^{-}$ while an incoming negatively charged scalar field may be identified with helicity $j^{-}\tilde{j}^{+}$. Any charge conservation-violating amplitude $\mathcal{A}$ must vanish. That is, any amplitude with unequal numbers of positively and negatively charged scalar fields will vanish. Let us start with a non-vanishing M-graviton scattering amplitude. Following the BDFS notation, let n_+ (n_-) be the number of " + " (" − ") helicities in YM amplitude A that have been flipped in YM amplitude $\tilde{A}$. Then the resulting amplitude vanishes whenever $n_+ \neq n_-$.

Let us consider the 4-point case to establish some notation: the graviton-dilaton-axion scattering amplitude takes the form

$$A_{\text{Einstein}} = -s_{12}A(1234)\tilde{A}(2134) \tag{51}$$

where both A and $\tilde{A}$ are YM amplitudes. For 4-graviton amplitudes, helicity conservation requires 2 with helicity $(++)$ and 2 with helicity $(--)$. So the only non-vanishing amplitude has the form

$$A_{\text{Einstein}} = -s_{12}A(1^-2^-3^+4^+)\tilde{A}(2^-1^-3^+4^+) \tag{52}$$

Note that both A and $\tilde{A}$ are maximal helicity-violating amplitudes. Next consider the 5-graviton scattering case,

$$\begin{aligned} A_{\text{Einstein},5} &= s_{12}s_{34}A(1^-2^-3^+4^+5^+)\tilde{A}(2^-1^-4^+3^+5^+) \\ &+ \; s_{13}s_{24}A(1^-3^+2^-4^+5^+)\tilde{A}(3^+1^-4^+2^-5^+) \end{aligned} \tag{53}$$

For $n_+ - n_- \neq 0$, the resulting $\mathcal{A}_5 = 0$. For example, for $(n_+, n_-) = (1,0)$, we have

$$\begin{aligned} 0 \; &= s_{12}s_{34}A(1^-2^-3^+4^+5^+)\tilde{A}(2^-1^-4^+3^-5^+) + \\ & s_{13}s_{24}A(1^-3^+2^-4^+5^+)\tilde{A}(3^-1^-4^+2^-5^+) \end{aligned} \tag{54}$$

It is easy to verify this quadratic identity by using the explicit formulae for the MHV amplitudes. For $M \geq 6$, non-MHV amplitudes appear in the quadratic identities.

III. FURTHER READING

For the tree-level and loop-level BCJ relation, see [3]. For the application of the BCJ relation for the five-loop $N = 8$ supergravity, check [4]. For the application of the BCJ relation for gravitational wave, check [4, 5] and the references therein.

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